利用克拉瑪公式

求兩歪斜線的公垂線段的兩端點座標的公式解法

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研究目的：試圖以克拉瑪公式來探求兩歪斜線的公垂線段的兩端點座標的公式解法。

研究過程：

已知空間直角座標系中，O爲原點，兩歪斜線L_1與L_2分別通過點$A_1\,(x_1,y_1,z_1)$、點$A_2\,(x_2,y_2,z_2)$，L_1與L_2的方向向量分別爲$\vec{d}_1=(l_1,m_1,n_1)$與$\vec{d}_2=(l_2,m_2,n_2)$，試求L_1與公垂線的交點B_1及L_2與公垂線的交點B_2的座標。

一、建立二元一次聯立方程組：

1. $\because B_1\in L_1,\because B_1=(x_1+l_1t_1,y_1+m_1t_1,z_1+n_1t_1)$，$t_1\in R$，

$\because B_2\in L_2,\because B_2=(x_2+l_2t_2,y_2+m_2t_2,z_2+n_2t_2)$，$t_2\in R$。

$\Rightarrow \overrightarrow{B_1B_2}=(x_2-x_1+l_2t_2-l_1t_1,y_2-y_1+m_2t_2-m_1t_1,z_2-z_1+n_2t_2-n_1t_1)$。

2.$\because \overrightarrow{B_1B_2}\perp\vec{d}_1$ 且 $\overrightarrow{B_1B_2}\perp\vec{d}_2$，

$\therefore \overrightarrow{B_1B_2}\cdot\vec{d}_1=0$ 且 $\overrightarrow{B_1B_2}\cdot\vec{d}_2=0$。

3. $$\begin{cases}0=\overrightarrow{B_1B_2}\cdot\vec{d}_1=(x_2-x_1+l_2t_2-l_1t_1)l_1+(y_2-y_1+m_2t_2-m_1t_1)m_1+(z_2-z_1+n_2t_2-n_1t_1)n_1\\0=\overrightarrow{B_1B_2}\cdot\vec{d}_2=(x_2-x_1+l_2t_2-l_1t_1)l_2+(y_2-y_1+m_2t_2-m_1t_1)m_2+(z_2-z_1+n_2t_2-n_1t_1)n_2\end{cases},$$

$$\Rightarrow\begin{cases}0=[(x_2-x_1)l_1+(y_2-y_1)m_1+(z_2-z_1)n_1]+t_2(l_1l_2+m_1m_2+n_1n_2)-t_1(l_1^2+m_1^2+n_1^2)\\0=[(x_2-x_1)l_2+(y_2-y_1)m_2+(z_2-z_1)n_2]+t_2(l_2^2+m_2^2+n_2^2)-t_1(l_1l_2+m_1m_2+n_1n_2)\end{cases},$$

$$\Rightarrow\begin{cases}0=\overrightarrow{A_1A_2}\cdot\vec{d}_1+t_2(\vec{d}_1\cdot\vec{d}_2)-t_1(\vec{d}_1\cdot\vec{d}_1)\\0=\overrightarrow{A_1A_2}\cdot\vec{d}_2+t_2(\vec{d}_2\cdot\vec{d}_2)-t_1(\vec{d}_1\cdot\vec{d}_2)\end{cases},$$

$$\Rightarrow\begin{cases}t_1(\vec{d}_1\cdot\vec{d}_1)-t_2(\vec{d}_1\cdot\vec{d}_2)=\overrightarrow{A_1A_2}\cdot\vec{d}_1\\t_1(\vec{d}_1\cdot\vec{d}_2)-t_2(\vec{d}_2\cdot\vec{d}_2)=\overrightarrow{A_1A_2}\cdot\vec{d}_2\end{cases}。$$

二、利用克拉瑪公式求解：

1.設$\vec{d}_1$與$\vec{d}_2$的夾角爲θ，$\because\vec{d}_1$與$\vec{d}_2$不平行$\because\sin\theta\neq0$。

2. $$\Delta=\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & -\vec{d}_1\cdot\vec{d}_2\\\vec{d}_1\cdot\vec{d}_2 & -\vec{d}_2\cdot\vec{d}_2\end{vmatrix}=-[|\vec{d}_1|^2|\vec{d}_2|^2-(\vec{d}_1\cdot\vec{d}_2)^2]=-|\vec{d}_1|^2|\vec{d}_2|^2\sin^2\theta\neq0,$$

由克拉瑪公式得知(t_1,t_2)恰有一組解。

$$t_1=\frac{\begin{vmatrix}\overrightarrow{A_1A_2}\cdot\vec{d}_1 & -\vec{d}_1\cdot\vec{d}_2\\ \overrightarrow{A_1A_2}\cdot\vec{d}_2 & -\vec{d}_2\cdot\vec{d}_2\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & -\vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & -\vec{d}_2\cdot\vec{d}_2\end{vmatrix}}=\frac{\begin{vmatrix}\overrightarrow{A_1A_2}\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \overrightarrow{A_1A_2}\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}\text{，}$$

$$t_2=\frac{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\overrightarrow{A_1A_2}\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\overrightarrow{A_1A_2}\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & -\vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & -\vec{d}_2\cdot\vec{d}_2\end{vmatrix}}=\frac{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\overrightarrow{A_2A_1}\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\overrightarrow{A_2A_1}\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}\text{。}$$

三、結論：

1. $$\overrightarrow{OB_1}=\overrightarrow{OA_1}+\overrightarrow{A_1B_1}=\overrightarrow{OA_1}+t_1\vec{d}_1=\overrightarrow{OA_1}+\frac{\begin{vmatrix}\overrightarrow{A_1A_2}\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \overrightarrow{A_1A_2}\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}\vec{d}_1\text{，}$$

2. $$\overrightarrow{OB_2}=\overrightarrow{OA_2}+\overrightarrow{A_2B_2}=\overrightarrow{OA_2}+t_2\vec{d}_2=\overrightarrow{OA_2}+\frac{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\overrightarrow{A_2A_1}\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\overrightarrow{A_2A_1}\end{vmatrix}}{\begin{vmatrix}\vec{d}_1\cdot\vec{d}_1 & \vec{d}_1\cdot\vec{d}_2\\ \vec{d}_1\cdot\vec{d}_2 & \vec{d}_2\cdot\vec{d}_2\end{vmatrix}}\vec{d}_2\text{。}$$

四、應用：

空間二歪斜線：$L_1:\dfrac{x-11}{4}=\dfrac{y+5}{-3}=\dfrac{z+7}{-1},L_2:\dfrac{x+5}{3}=\dfrac{y-4}{-4}=\dfrac{z-6}{-2}$，

$\overrightarrow{d_1}=(4,-3-1),\overrightarrow{d_2}=(3,-4,-2),\overrightarrow{OA_1}=(11,-5,-7),\overrightarrow{OA_2}=(-5,4,6)$，$O$ 爲空間

直角坐標系的原點，$\overrightarrow{A_1A_2}=(-16,9,13)$。試求：

(1) L_1 與公垂線的交點 B_1。
(2) L_2 與公垂線的交點 B_2。
解：

$$t_1=\frac{\begin{vmatrix}-104 & 26\\ -110 & 29\end{vmatrix}}{\begin{vmatrix}26 & 26\\ 26 & 29\end{vmatrix}}=\frac{\begin{vmatrix}-52 & 26\\ -52 & 29\end{vmatrix}}{\begin{vmatrix}26 & 26\\ 26 & 29\end{vmatrix}}=-2,t_2=\frac{\begin{vmatrix}26 & 104\\ 26 & 110\end{vmatrix}}{\begin{vmatrix}26 & 26\\ 26 & 29\end{vmatrix}}=\frac{\begin{vmatrix}26 & 52\\ 26 & 58\end{vmatrix}}{\begin{vmatrix}26 & 26\\ 26 & 29\end{vmatrix}}=2$$

$\overrightarrow{OB_1}=(11,-5,-7)+(-2)\cdot(4,-3,-1)=(3,1,-5)$，

$\overrightarrow{OB_2}=(-5,4,6)+2\cdot(3,-4,-2)=(1,-4,2)$。

註：本文所有的射線符號” $\vec{}$ ”都代表向量符號。